Solution for 1:

Proof. Let $P(n)$ be the statement "The current player can guarantee a win if $n = 5a$ for some $a \in \mathbb{Z}^+$." We will prove $P(n)$, $\forall n \in \mathbb{Z}^+$.

First note that if $n \neq 5a$ for some $a \in \mathbb{Z}^+$, then $P(n)$ is vacuously true. Thus, if we show that $P(n)$ holds for all multiples of 5, then we can conclude $P(n)$ holds $\forall n \in \mathbb{Z}^+$.

For the base case $n = 5$, there are 5 cookies and the current player can guarantee a win if they choose to eat 4 of them, forcing the opponent to eat the last cookie. Thus $P(5)$ holds.

Now suppose for some $k \in \mathbb{Z}^+$ such that $k = 5a$ (for some $a \in \mathbb{Z}^+$), $P(k)$ holds. In words, suppose the current player can guarantee a win if $k = 5a$ for some $a \in \mathbb{Z}^+$. We want to show $P(k+5)$.

Since there are $k+5$ cookies where $k \geq 5$, the current player should eat 4 cookies, leaving $k+1$ cookies for the opponent. We have two cases to consider:

- If the opponent eats 1 cookie, then there are k cookies left and the current player wins by the induction hypothesis.
- If the opponent eats m cookies, where $2 \leq m \leq 4$, then the current player must eat $5 - m$ cookies. Note that $-m - (5 - m) = 5$. So in total, the opponent's turn and the current player's turn will always subtract 5 from the remaining cookies, leaving $k - 4 \equiv 1 \mod 5$. This process will continue until the opponent is left with $5 + 1 = 6$ cookies ($6 \equiv 1 \mod 5$):
  - If the opponent eats 1, then there will be 5 cookies left, and the current player will win by the base case.
  - If the opponent eats 2, then there will be 4 cookies left, and the current player can eat 3 to guarantee a win.
  - If the opponent eats 3, then there will be 3 cookies left, and the current player can eat 2 to guarantee a win.
  - If the opponent eats 4, then there will be 2 cookies left, and current player can eat 1 to guarantee a win.

Thus, the current player can guarantee a win if there are $k + 5$ cookies.

Therefore, by the principal of simple induction, $P(n)$ is true $\forall n \in \mathbb{Z}^+$ as required.

$\square$

Solution for 2:
If there are $n = 24$ cookies, it is better to go first (Player 1). Your first move should be to eat 3 cookies to leave the other player with 21 cookies.

If the opponent eats m cookies for $1 \leq m \leq 4$, Player 1 must eat $5 - m$ cookies so that the opponent is left with 16 cookies. As an example, at 21 cookies, if the other player chooses to eat 4, you should eat 1.

If the opponent eats m cookies for $1 \leq m \leq 4$, Player 1 must eat $5 - m$ cookies so that the opponent is left with 11 cookies.

If the opponent eats m cookies for $1 \leq m \leq 4$, Player 1 must eat $5 - m$ cookies so that the opponent is left with 6 cookies.

If the opponent eats m cookies for $1 \leq m \leq 4$, Player 1 must eat $5 - m$ cookies so that the opponent is left with 1 cookie.

The opponent will have no other choice but to eat the last cookie, losing the game to Player 1.

Therefore, it is better to go first as long as the player uses the strategy above.

Solution for 3:
The optimal strategy to win the game is to put the other player in a position where there are $n = 5a+1$ (where $a \in \mathbb{N}$) cookies remaining and to work with them at removing 5 cookies every two turns until they have to eat the last cookie. Let $L = \{1, 6, 11, \dots\}$ be the set of losing positions. So $n \in L \iff n = 5a + 1, a \in \mathbb{N}$. Notice that when $n = 5a + 1$, the number of cookies divided by 5 give a remainder of 1. This is because we do not want the other player to reach $n = 5a$ (where $a \in \mathbb{Z}^+$) otherwise they can guarantee a win by **Part 1**. When there are $n = 5a + 1$ cookies left for them, we can always put the opponent back in a similar position (by helping at removing 5 cookies) until they eat the last cookie. In other words, it takes two turns to get to a losing position. This strategy was demonstrated in **Part 2**, where we decided to go first since $24 \equiv 4 \mod 5$ (i.e. when 24 is divided by 5 its remainder is **not** 1). So we can move the other player to the losing position $21 = 5 \cdot 4 + 1$ by eating 3. Now no matter if the other player eats $1, 2, 3,$ or 4, we can eat in such a way we put them at $16 = 5 \cdot 3 + 1$. **In other words, we can eat in such a way to work with the other player at removing 5 cookies so that they stay in the losing positions!** This works because the other player is only allowed to eat between $1 - 4$ cookies (inclusive) and the losing positions differ by 5. So we have the final say when it comes to who stays in the losing positions! If you manage to get stuck in a losing position, then it is almost impossible to get out if the other player knows this strategy!