Solution:
Note: Before we proceed to the solution, let us define r_1, r_2, r_3 to be three rods lined up in a row with r_2 in the centre. Let $d_1, d_2, \ldots, d_n$ be n disks in increasing size order. This means d_1 is the smallest disk and d_n is the largest disk. Let these disks start off on r_1. The bottom disk on r_1 is d_n and the other disks are placed on top d_n in decreasing order (this means d_1 is at the top of r_1).

Claim: If n is odd, then to complete the Tower of Hanoi game in $2^n - 1$ moves, the first move should be to move d_1 to r_3. If n is even, then to complete the Tower of Hanoi game in $2^n - 1$ moves, the first move should be to move d_1 to r_2.

Proof (Odd). Let n be an odd positive integer. Let $P(n)$ be the statement "move d_1 to r_3 to complete the Tower of Hanoi puzzle in $2^n - 1$ moves. We will prove $P(n)$ for all odd positive integers (that is, we are proving $P(n)$ for any odd number of disks).

Let us begin by first verifying the base case $n = 1$. There is only one disk, d_1. Simply move it to r_3. This completes the puzzle in one move. Also, $2^1 - 1 = 1$. Thus, $P(1)$ holds.

Let n be an positive odd integer and suppose for all positive odd integers k between $1 \leq k \leq n$, $P(k)$ holds. We want to show $P(n+2)$ is true.

Begin by making a partition of the $k+2$ disks. Treat $d_1, d_2, \ldots, d_k$ as a single disk d and we have two other disks d_{k+1}, d_{k+2}. Now the problem reduces to moving three disks d_{k+2}, d_{k+1}, d to r_3. By the induction hypothesis, the best first move for three disks is to move d to r_3. So we have the following,

1. Move d to r_3. This will take $2^k - 1$ moves.
2. Then move d_{k+1} to r_2. This will take 1 move.
3. Then move d to r_2. This will take $2^k - 1$ moves.
4. Then move d_{k+2} to r_3. This will take 1 move.
5. Then move d to r_1. This will take $2^k - 1$ moves.
6. Then move d_{k+1} to r_3. This will take 1 move.
7. Finally, move d_{k+1} to r_3. This will take $2^k - 1$ moves.

Thus, in total, we have

$$\begin{aligned} 2^k - 1 + 1 + 2^k - 1 + 1 + 2^k - 1 + 1 + 2^k - 1 &= 4(2^k - 1) + 3 \\ &= 2^2 \cdot 2^k - 4 + 3 \\ &= 2^{k+2} - 1 \end{aligned}$$

This proves $P(n+2)$.

Therefore, by complete induction, the best first move when an odd number of disks are present is to move the smallest disk to the destination rod. □

Proof (Even). Let n be an even positive integer. Let $P(n)$ be the statement "move d_1 to r_2 to complete the Tower of Hanoi puzzle in $2^n - 1$ moves. We will prove $P(n)$ for all even positive integers (that is, we are proving $P(n)$ for any even number of disks).

Let us begin by first verifying the base case $n = 2$. There are only two disks, d_1, d_2. First move d_1 to r_2. Then move d_2 to r_3. Finally, move d_1 to r_3. This completes the puzzle in three moves. Also, $2^2 - 1 = 3$. Thus, $P(2)$ holds.

Let n be an positive even integer and suppose for all positive even integers k between $2 \leq k \leq n$, $P(k)$ holds. We want to show $P(n + 2)$ is true.

Begin by making a partition of the $k+2$ disks. Treat $d_1, d_2, \ldots, d_k, d_{k+1}$ as a single disk d and we have one other disk d_{k+2}. Now the problem reduces to moving two disks d_{k+2}, d to r_3. By the induction hypothesis, the best first move for 2 disks is to move d to r_2. So we have the following,

1. By the induction hypothesis, move d to r_2. This will take $2^{k+1} - 1$ moves.

2. Then move d_{k+2} to r_3. This will take 1 move.

3. Finally, move d to r_3. This will take $2^k - 1$ moves.

Thus, in total, we have

$$\begin{aligned} 2^{k+1} - 1 + 1 + 2^{k+1} - 1 &= 2(2^{k+1} - 1) + 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

This proves $P(k + 2)$.

Therefore, by complete induction, the best first move when an even number of disks are present is to move the smallest disk to the middle rod. $\square$