Proof. To prove $f(\mathbb{R}) = [-\frac{1}{2}, \frac{1}{2}]$, we need to show $f(\mathbb{R}) \subseteq [-\frac{1}{2}, \frac{1}{2}]$ and $[-\frac{1}{2}, \frac{1}{2}] \subseteq f(\mathbb{R})$.
To show $f(\mathbb{R}) \subseteq [-\frac{1}{2}, \frac{1}{2}]$, pick $y \in f(\mathbb{R})$ and argue $y \in [-\frac{1}{2}, \frac{1}{2}]$. Since $y \in f(\mathbb{R})$, it follows that $y = f(x) = \frac{x}{1+x^2}$ for some $x \in \mathbb{R}$.
Recall, $\forall a \in \mathbb{R}$ we have $a^2 \geq 0$. Hence, for any real number x, we have $-(x+1)^2 \leq 0$ and $(x-1)^2 \geq 0$. Combining the two inequalities and rearranging

$$\begin{aligned}
-(x+1)^2 \leq 0 &\leq (x-1)^2 \\
-(x^2+2x+1) \leq 0 &\leq x^2 - 2x + 1 \\
-x^2 - 2x - 1 \leq 0 &\leq x^2 - 2x + 1 \\
-x^2 - 1 \leq 2x &\leq x^2 + 1 \\
-(x^2+1) \leq 2x &\leq x^2 + 1 \\
-1 \leq \frac{2x}{x^2+1} &\leq 1 \qquad \text{(since } x^2+1>0\text{)} \\
-\frac{1}{2} \leq \frac{x}{1+x^2} &\leq \frac{1}{2} \\
-\frac{1}{2} \leq y &\leq \frac{1}{2}
\end{aligned}$$

shows that $y \in [-\frac{1}{2}, \frac{1}{2}]$. Therefore, $f(\mathbb{R}) \subseteq [-\frac{1}{2}, \frac{1}{2}]$.
To show $[-\frac{1}{2}, \frac{1}{2}] \subseteq f(\mathbb{R})$, pick $y \in [-\frac{1}{2}, \frac{1}{2}]$ and argue $y \in f(\mathbb{R})$ (which means that we need to find an input $x \in \mathbb{R}$ such that $f(x) = y$).
If $y = 0$ then pick $x = 0 \in \mathbb{R}$, and observe that $f(x) = f(0) = \frac{0}{1+0} = 0 = y$ which shows that $y \in f(\mathbb{R})$.
If $y \neq 0$ then pick $x = \frac{1+\sqrt{1-4y^2}}{2y} \in \mathbb{R}$ since $y \in [-\frac{1}{2}, \frac{1}{2}] \backslash \{0\}$. Observe that

$$\begin{aligned}
f(x) &= f\left(\frac{1+\sqrt{1-4y^2}}{2y}\right) \\
&= \frac{\frac{1+\sqrt{1-4y^2}}{2y}}{1+\left(\frac{1+\sqrt{1-4y^2}}{2y}\right)^2} \\
&= \frac{\frac{1+\sqrt{1-4y^2}}{2y}}{\frac{4y^2+\left(1+\sqrt{1-4y^2}\right)^2}{4y^2}} \\
&= \frac{1+\sqrt{1-4y^2}}{2y} \cdot \left(\frac{4y^2}{4y^2+\left(1+\sqrt{1-4y^2}\right)^2}\right) \\
&= \frac{2y\left(1+\sqrt{1-4y^2}\right)}{4y^2+\left(1+\sqrt{1-4y^2}\right)^2} \\
&= \frac{2y\left(1+\sqrt{1-4y^2}\right)}{4y^2+\left(1+2\sqrt{1-4y^2}+1-4y^2\right)} \\
&= \frac{2y\left(1+\sqrt{1-4y^2}\right)}{2+2\sqrt{1-4y^2}} \\
&= \frac{2y\left(1+\sqrt{1-4y^2}\right)}{2\left(1+\sqrt{1-4y^2}\right)} \\
&= y
\end{aligned}$$

which shows that $y \in f(\mathbb{R})$.
Overall we have that $f(\mathbb{R}) \subseteq [-\frac{1}{2}, \frac{1}{2}]$.

Therefore $f(\mathbb{R}) = [-\frac{1}{2}, \frac{1}{2}]$, as required. □