

Part 1 Solution:

The distance between the mark left at the scene and the next mark is 70π cm. This implies that the wheel completed one full revolution, and so the circumference of the wheel is 70π cm. We can find the diameter of the wheel using $C = \pi \cdot d$.

$$\begin{aligned} 70\pi &= \pi \cdot d \\ d &= 70 \end{aligned}$$

Therefore, the diameter of the wheel is 70 cm.

Part 2 Solution:

Let us analyze the scene through the eyes of the witness.

The wheel completed two full revolutions, and on the third revolution the button was 10 cm above the window before the witness lost sight of it.

The two full revolutions correspond to a distance of $70\pi + 70\pi = 140\pi$ cm. Now let us analyze the third revolution. When the button was 10 cm above the window, its height above ground is $10 + 50 = 60$ cm. So the button's position is given by $P(x, 60)$. Our goal is to find the x -coordinate of P which represents the horizontal distance and add it to 140π cm.

To find the x -coordinate, we will use the parametric equations of a cycloid

$$x = r(\theta - \sin \theta) \quad y = r(1 - \cos \theta)$$

where x represents the horizontal displacement, y represents the vertical displacement, r represents the radius of the wheel, and θ represents the angle the wheel has rotated.

Since we have the vertical displacement ($y = 60$ cm), we can use it to find the angle the wheel has rotated using $y = r(1 - \cos \theta)$.

$$\begin{aligned} 60 &= 35(1 - \cos \theta) \\ \frac{12}{7} &= 1 - \cos \theta \\ \cos \theta &= -\frac{5}{7} \end{aligned}$$

The reference angle is given by $\theta_r = \cos^{-1}\left(\frac{5}{7}\right)$. Since $\cos \theta < 0$ we have an angle in quadrant II and an angle in quadrant III.

$$\begin{aligned} \theta_1 &= \pi - \theta_r = \pi - \cos^{-1}\left(\frac{5}{7}\right) \\ \theta_2 &= \pi + \theta_r = \pi + \cos^{-1}\left(\frac{5}{7}\right) \end{aligned}$$

θ_1 represents the first time the button reaches a height of 60 cm and so we are interested in θ_2 . Let $\theta = \theta_2 = \pi + \cos^{-1}\left(\frac{5}{7}\right)$.

Now that we found θ , we can find the horizontal displacement using $x = r(\theta - \sin \theta)$.

$$\begin{aligned}
 x &= 35 \left(\pi + \cos^{-1} \left(\frac{5}{7} \right) - \sin \left(\pi + \cos^{-1} \left(\frac{5}{7} \right) \right) \right) \\
 &= 35\pi + 35 \cos^{-1} \left(\frac{5}{7} \right) - 35 \sin \left(\pi + \cos^{-1} \left(\frac{5}{7} \right) \right) \\
 &= 35\pi + 35 \cos^{-1} \left(\frac{5}{7} \right) - 35 \left(\sin(\pi) \cos \left(\cos^{-1} \left(\frac{5}{7} \right) \right) + \cos(\pi) \sin \left(\cos^{-1} \left(\frac{5}{7} \right) \right) \right) \\
 & \hspace{20em} \text{(used angle-sum formula)} \\
 &= 35\pi + 35 \cos^{-1} \left(\frac{5}{7} \right) - 35 \left(0 \cdot \frac{5}{7} + (-1) \left(\frac{2\sqrt{6}}{7} \right) \right) \hspace{5em} \text{(Aside)} \\
 &= 35\pi + 35 \cos^{-1} \left(\frac{5}{7} \right) - 35 \left(-\frac{2\sqrt{6}}{7} \right) \\
 &= 35\pi + 35 \cos^{-1} \left(\frac{5}{7} \right) + 10\sqrt{6}
 \end{aligned}$$

Now we can find the total distance,

$$\begin{aligned}
 d_T &= 140\pi + 35\pi + 35 \cos^{-1} \left(\frac{5}{7} \right) + 10\sqrt{6} \\
 &= 175\pi + 10\sqrt{6} + 35 \cos^{-1} \left(\frac{5}{7} \right)
 \end{aligned}$$

Therefore, the suspect was $\left[175\pi + 10\sqrt{6} + 35 \cos^{-1} \left(\frac{5}{7} \right) \right]$ cm from the scene when the witness saw the button for the last time.

Aside: To find $\sin \left(\cos^{-1} \left(\frac{5}{7} \right) \right)$, let $\phi = \cos^{-1} \left(\frac{5}{7} \right)$. Then $\cos(\phi) = \frac{5}{7}$. From this you can create a right angle triangle and determine that $\sin(\phi) = \frac{2\sqrt{6}}{7}$.