**Solution:**
To find a vector equation for $L_2$ we need $\vec{d} = \overrightarrow{AB}$ (direction vector) and $P_0 = A$ (point on $L_2$).

$$\vec{d} = (1-1, 2-1, 3-1) = (0,1,2) \text{ and } P_0 = (1,1,1)$$

Thus, $L_2$ has vector equation $\vec{r}(t) = (1,1,1) + t(0,1,2)$.

The point $P_1 = (1+s, s, 0)$ represents a general point on $L_1$.
The point $P_2 = (1, 1+t, 1+2t)$ represents a general point on $L_2$.
Next we find $\overrightarrow{P_1P_2}$ which is a general vector whose tail lies on $L_1$ and head on $L_2$.

$$\overrightarrow{P_1P_2} = (1-(1+s), (1+t)-s, (1+2t)-0) = (-s, 1+t-s, 1+2t)$$

The points on $L_1$ and $L_2$ that produce the minimum distance must satisfy $\overrightarrow{P_1P_2}\cdot\vec{d_1} = 0$ and $\overrightarrow{P_1P_2}\cdot\vec{d_2} = 0$. This is because $\overrightarrow{P_1P_2}$ must be perpendicular to each line for shortest distance.

$$\begin{aligned} \overrightarrow{P_1P_2}\cdot\vec{d_1} &= 0 \\ (-s, 1+t-s, 1+2t)\cdot(1,1,0) &= 0 \\ -s+1+t-s &= 0 \\ -2s+t &= -1 \end{aligned} \qquad \begin{aligned} \overrightarrow{P_1P_2}\cdot\vec{d_2} &= 0 \\ (-s, 1+t-s, 1+2t)\cdot(0,1,2) &= 0 \\ 1+t-s+2+4t &= 0 \\ -s+5t &= -3 \end{aligned}$$

Now we solve the following augmented matrix

$$\left(\begin{array}{cc|c} -2 & 1 & -1 \\ -1 & 5 & -3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & \frac{2}{9} \\ 0 & 1 & -\frac{5}{9} \end{array}\right)$$

Thus, $s = \frac{2}{9}$ and $t = -\frac{5}{9}$.
Now substitute $s = \frac{2}{9}$ and $t = -\frac{5}{9}$ into $L_1$ and $L_2$ to find the required points.

$$L_1 : \vec{p}\left(\frac{2}{9}\right) = (1,0,0) + \frac{2}{9}(1,1,0) = \left(\frac{11}{9}, \frac{2}{9}, 0\right)$$

$$L_2 : \vec{r}\left(-\frac{5}{9}\right) = (1,1,1) - \frac{5}{9}(0,1,2) = \left(1, \frac{4}{9}, -\frac{1}{9}\right)$$

Therefore, $P_1 = \left(\frac{11}{9}, \frac{2}{9}, 0\right)$ and $P_2 = \left(1, \frac{4}{9}, -\frac{1}{9}\right)$ are the points on $L_1$ and $L_2$, respectively, that are closest to together.

**Extra:** The shortest distance between the two lines is given by $\|\overrightarrow{P_1P_2}\|$.

$$\overrightarrow{P_1P_2} = \left(1 - \frac{11}{9}, \frac{4}{9} - \frac{2}{9}, -\frac{1}{9} - 0\right) = \left(-\frac{2}{9}, \frac{2}{9}, -\frac{1}{9}\right)$$

$$\|\overrightarrow{P_1P_2}\| = \sqrt{\left(-\frac{2}{9}\right)^2 + \left(\frac{2}{9}\right)^2 + \left(-\frac{1}{9}\right)^2} = \frac{1}{3}$$

Therefore, the shortest distance between $L_1$ and $L_2$ is $\frac{1}{3}$.