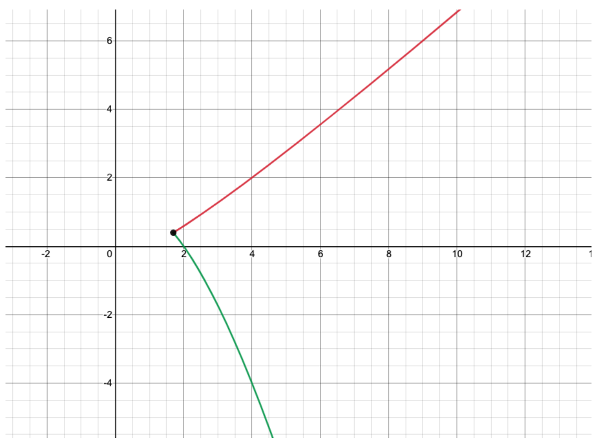
# Problem 1: A Golden Intersection

## Part A

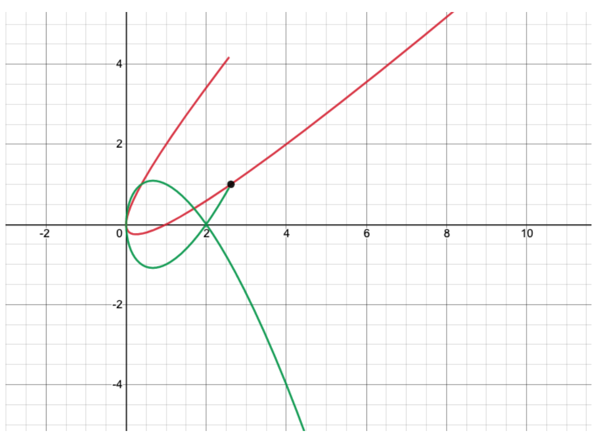
The flaw in Tonga's reasoning occurs when she sets $\alpha(t) = \beta(t)$. This statement assumes that all the points of intersection occur at the same value of $t$. If we assume $t$ represents time, then Tonga found the times where the curves are at the same point. In other words, the curves need to collide in order for them to intersect. With this way of thinking, Tonga could have missed some points of intersection since points of intersection do not necessarily have to occur at the same values of $t$ for both curves (i.e a curve can pass through the same point as the other graph at a later time). A better approach would be to differentiate between the two $t$'s and solve:

$$\alpha(t_1) = \beta(t_2)$$

To illustrate my point, consider the following diagrams ($\alpha(t)$ is the red curve and $\beta(t)$ is the green curve):



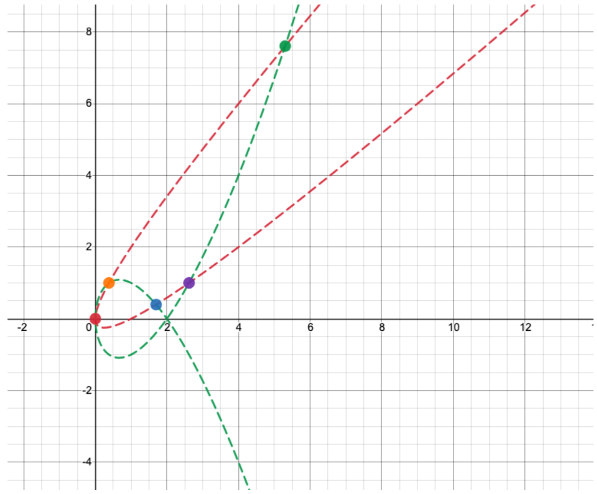

Here both curves collide. In other words, both curves pass through the same point at the same time (same value of t). As a result, Tonga was able to find this value ($t = \frac{1-\sqrt{13}}{2}$).




Here both curves intersect each other, but for different values of $t$. Said differently, both curves pass through the same point, but at different times. As a result, Tonga was not able to find this value as a solution.

## Part B

The curves intersect a total of 5 times by the following diagram:



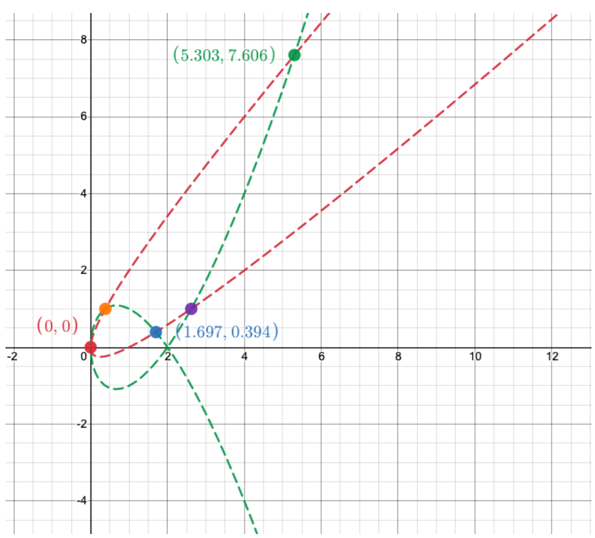
Thanks to Tonga we already know when three of them occur, so our job is to first find the Cartesian coordinates of those three points and then work on finding the last two.
Since Tonga found that the curves collide at $t = 0, \frac{1-\sqrt{13}}{2}, \frac{1+\sqrt{13}}{2}$ we can plug these $t$ values into either $\alpha(t)$ or $\beta(t)$ to get the Cartesian coordinates:

$$\alpha(0) = (0, 0)$$

$$\alpha\left(\frac{1-\sqrt{13}}{2}\right) = \left(\frac{7-\sqrt{13}}{2}, 4-\sqrt{13}\right)$$

$$\alpha\left(\frac{1+\sqrt{13}}{2}\right) = \left(\frac{7+\sqrt{13}}{2}, 4+\sqrt{13}\right)$$

The following diagram shows the three points we just found.

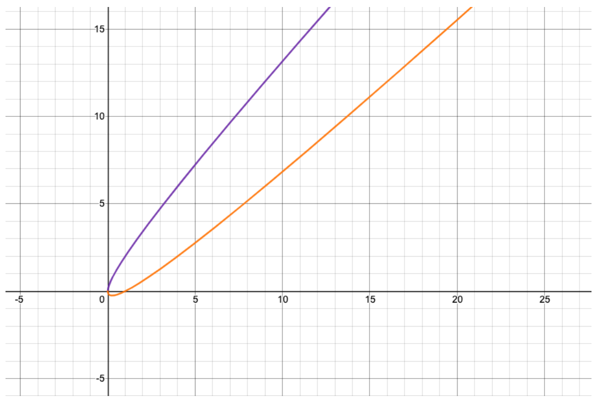


By the above diagram, we now need to find the orange and purple points. Let the orange point be $A$ and the purple point be $B$. To find these points, we will convert $\alpha(t)$ and $\beta(t)$ to Cartesian form. We will probably need to express them as multiple functions of the form $y = f(x)$.

Converting $\alpha(t)$ to Cartesian form:
From $\alpha(t) = (t^2, t + t^2)$, we know $x = t^2$ and $y = t + t^2$. Thus, $t = \pm\sqrt{x}$ and plugging into $y$, we get $y = \pm\sqrt{x} + x$. So $\alpha(t)$ can be expressed by

$$\begin{cases} f_1(x) = \sqrt{x} + x \\ f_2(x) = -\sqrt{x} + x \end{cases}$$

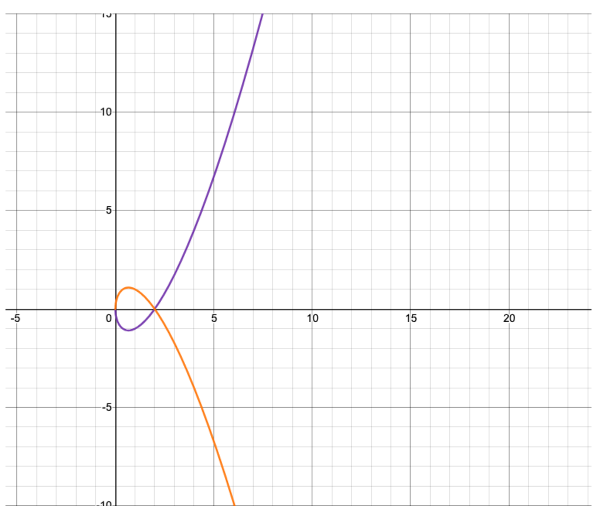
The following diagram shows the two pieces of $\alpha(t)$ ($f_1(x)$ is the purple curve and $f_2(x)$ is the orange curve).




Converting $\beta(t)$ to Cartesian form:
From $\beta(t) = (t^2, t^3 - 2t)$, we know $x = t^2$ and $y = t^3 - 2t$. Thus, $t = \pm\sqrt{x}$ and plugging into $y$, we get $y = (\pm\sqrt{x})^3 - 2(\pm\sqrt{x})$. So $\beta(t)$ can be expressed by

$$\begin{cases} g_1(x) = x^{3/2} - 2\sqrt{x} \\ g_2(x) = -x^{3/2} + 2\sqrt{x} \end{cases}$$

The following diagram shows the two pieces of $\beta(t)$ ($g_1(x)$ is the purple curve and $g_2(x)$ is the orange curve).



Now looking at the above diagrams that show all the intersection points, we can see that to find the point $A$ we need to find the intersection of $f_1(x)$ and $g_2(x)$, and to find the point $B$ we need to find the intersection of $f_2(x)$ and $g_1(x)$.

Intersection of $f_1(x)$ and $g_2(x)$:

$$f_1(x) = g_2(x)$$
$$\sqrt{x} + x = -x^{3/2} + 2\sqrt{x}$$
$$x^{3/2} - \sqrt{x} + x = 0$$

Let $m = \sqrt{x}$

$$m^3 - m + m^2 = 0$$
$$m(m^2 + m - 1) = 0$$
$$m = 0 \vee m = \frac{-1 \pm \sqrt{5}}{2}$$

Since $x = m^2$, we have

$$x = 0 \vee x = \frac{3 \pm \sqrt{5}}{2}$$

We already found the solution associated with $x = 0$. Moreover, $x = \frac{3+\sqrt{5}}{2}$ does not satisfy the equation $x^{3/2} - \sqrt{x} + x = 0$ so it is not a solution. The only solution is $x = \frac{3-\sqrt{5}}{2}$ which corresponds to a y-value of

$$y = f_1\left(\frac{3-\sqrt{5}}{2}\right) = \sqrt{\left(\frac{3-\sqrt{5}}{2}\right) + \left(\frac{3-\sqrt{5}}{2}\right)} = 1$$

.
Thus, the intersection point in Cartesian coordinates is $\left(\frac{3-\sqrt{5}}{2}, 1\right)$.

Intersection of $f_2(x)$ and $g_1(x)$:

$$f_2(x) = g_1(x)$$
$$-\sqrt{x} + x = x^{3/2} - 2\sqrt{x}$$
$$x^{3/2} - \sqrt{x} - x = 0$$

Let $v = \sqrt{x}$

$$v^3 - v - v^2 = 0$$
$$v(v^2 - v - 1) = 0$$
$$v = 0 \vee v = \frac{1 \pm \sqrt{5}}{2}$$

Since $x = v^2$, we have

$$x = 0 \vee x = \frac{3 \pm \sqrt{5}}{2}$$

We already found the solution associated with $x = 0$. Moreover, $x = \frac{3-\sqrt{5}}{2}$ does not satisfy the equation $x^{3/2} - \sqrt{x} - x = 0$ so it is not a solution. The only solution is $x = \frac{3+\sqrt{5}}{2}$ which corresponds to a y-value of

$$y = f_2\left(\frac{3+\sqrt{5}}{2}\right) = -\sqrt{\left(\frac{3+\sqrt{5}}{2}\right) + \left(\frac{3+\sqrt{5}}{2}\right)} = 1$$

.

Thus, the intersection point in Cartesian coordinates is $\left(\frac{3+\sqrt{5}}{2}, 1\right)$.

Overall, the 5 points of intersection are:

$$(0,0), \left(\frac{7-\sqrt{13}}{2}, 4-\sqrt{13}\right), \left(\frac{7+\sqrt{13}}{2}, 4+\sqrt{13}\right), \left(\frac{3-\sqrt{5}}{2}, 1\right), \left(\frac{3+\sqrt{5}}{2}, 1\right)$$

The intersection points are shown below: