Problem 2: Ellipse Twins

Part A

To find a Cartesian equation for E_2, we will first convert E_1 to polar form. Then apply the clockwise $\frac{\pi}{4}$ transformation to it. Then convert it back to Cartesian form.

E_1 is defined to be $x^2 + xy + y^2 = 1$. To convert E_1 to polar, we will use $y = r\sin\theta$ and $x = r\cos\theta$. Substituting these into E_1 and simplifying:

$$
\begin{aligned}
&(r\cos\theta)^2 + (r\cos\theta)(r\sin\theta) + (r\sin\theta)^2 = 1 \\
&\implies r^2\cos^2\theta + r^2\cos\theta\sin\theta + r^2\sin^2\theta = 1 \\
&\implies r^2(\cos^2\theta + \sin^2\theta + \cos\theta\sin\theta) = 1 \\
&\implies r^2(1 + \cos\theta\sin\theta) = 1 && (\text{since } \cos^2\theta + \sin^2\theta = 1) \\
&\implies r^2\left(1 + \frac{\sin(2\theta)}{2}\right) = 1 && (\text{since } \cos\theta\sin\theta = \frac{\sin(2\theta)}{2}) \\
&\implies r^2 = \frac{2}{2 + \sin(2\theta)}
\end{aligned}
$$

Now that we have E_1 in polar form, we can apply the clockwise $\frac{\pi}{4}$ transformation:

$$r^2 = \frac{2}{2 + \sin(2(\theta + \frac{\pi}{4}))}$$

This is E_2. Now we just convert it to Cartesian form using $r^2 = x^2 + y^2$, $y = r\sin\theta$, and $x = r\cos\theta$:

$$
\begin{aligned}
&r^2 = \frac{2}{2 + \sin(2\theta + \frac{\pi}{2})} \\
&\implies r^2 = \frac{2}{2 + \cos(2\theta)} && (\text{since } \sin\left(x + \frac{\pi}{2}\right) = \cos(x)) \\
&\implies r^2(2 + \cos(2\theta)) = 2 \\
&\implies r^2(2 + \cos^2\theta - \sin^2\theta) = 2 && (\text{since } \cos(2x) = \cos^2 x - \sin^2 x) \\
&\implies 2r^2 + r^2\cos^2\theta - r^2\sin^2\theta = 2 \\
&\implies 2(x^2 + y^2) + x^2 - y^2 = 2 \\
&\implies 3x^2 + y^2 = 2
\end{aligned}
$$

Thus, the Cartesian equation of E_2 is $3x^2 + y^2 = 2$ as required.

Part B

Now to find P and Q, we will make use of the polar equations of E_1 and E_2.
Recall the following:

$$E_1 : r^2 = \frac{2}{2 + \sin(2\theta)}, \quad E_2 : r^2 = \frac{2}{2 + \sin(2\theta + \frac{\pi}{2})}$$

Let us equate them to find P and Q as follows,

$$\begin{aligned}
\frac{2}{2+\sin(2\theta)} &= \frac{2}{2+\sin(2\theta+\frac{\pi}{2})} \\
2(2+\cos(2\theta)) &= 2(2+\sin(2\theta)) && \left(\text{since } \sin\left(x+\frac{\pi}{2}\right)=\cos(x)\right) \\
\cos(2\theta) &= \sin(2\theta) \\
\tan(2\theta) &= 1 \\
2\theta &= \frac{\pi}{4}+\pi k, k\in\mathbb{Z} \\
\theta &= \frac{\pi}{8}+\frac{\pi}{2}k, k\in\mathbb{Z}
\end{aligned}$$

If $\theta \in [0, 2\pi]$, then

$$\theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

Since P and Q are in the first and second quadrants, the angle for P will correspond to $\theta = \frac{\pi}{8}$ and angle for Q will correspond to $\theta = \frac{5\pi}{8}$. To get the Cartesian coordinates, we first find the polar coordinates by finding the value of r for each angle. We can use the polar form of either E_1 or E_2:

For $\theta = \frac{\pi}{8}$,

$$r^2 = \frac{2}{2+\sin(2(\frac{\pi}{8}))} \implies r = \sqrt{\frac{8-2\sqrt{2}}{7}}$$

For $\theta = \frac{5\pi}{8}$,

$$r^2 = \frac{2}{2+\sin(2(\frac{5\pi}{8}))} \implies r = \sqrt{\frac{8+2\sqrt{2}}{7}}$$

So the polar coordinates of P are $\left(\sqrt{\frac{8-2\sqrt{2}}{7}}, \frac{\pi}{8}\right)$ and the polar coordinates of Q are $\left(\sqrt{\frac{8+2\sqrt{2}}{7}}, \frac{5\pi}{8}\right)$. Now all we need to do is convert $\left(\sqrt{\frac{8-2\sqrt{2}}{7}}, \frac{\pi}{8}\right)$ and $\left(\sqrt{\frac{8+2\sqrt{2}}{7}}, \frac{5\pi}{8}\right)$ to Cartesian coordinates using $(x, y) = (r\cos\theta, r\sin\theta)$.

$$\left(\sqrt{\frac{8-2\sqrt{2}}{7}}, \frac{\pi}{8}\right) \to \left(\sqrt{\frac{8-2\sqrt{2}}{7}}\cos\left(\frac{\pi}{8}\right), \sqrt{\frac{8-2\sqrt{2}}{7}}\sin\left(\frac{\pi}{8}\right)\right)$$

$$\left(\sqrt{\frac{8+2\sqrt{2}}{7}}, \frac{5\pi}{8}\right) \to \left(\sqrt{\frac{8+2\sqrt{2}}{7}}\cos\left(\frac{5\pi}{8}\right), \sqrt{\frac{8+2\sqrt{2}}{7}}\sin\left(\frac{5\pi}{8}\right)\right)$$

Thus, the Cartesian coordinates of P are $\left(\sqrt{\frac{8-2\sqrt{2}}{7}}\cos\left(\frac{\pi}{8}\right), \sqrt{\frac{8-2\sqrt{2}}{7}}\sin\left(\frac{\pi}{8}\right)\right)$ and the Cartesian coordinates of Q are $\left(\sqrt{\frac{8+2\sqrt{2}}{7}}\cos\left(\frac{5\pi}{8}\right), \sqrt{\frac{8+2\sqrt{2}}{7}}\sin\left(\frac{5\pi}{8}\right)\right)$ as required.

Part C

We already found the polar coordinates of P and Q in the previous part. The polar coordinates of P are $\left(\sqrt{\frac{8-2\sqrt{2}}{7}}, \frac{\pi}{8}\right)$ and the polar coordinates of Q are $\left(\sqrt{\frac{8+2\sqrt{2}}{7}}, \frac{5\pi}{8}\right)$.

Part D

There are many reasons why using polar integration with "outer minus inner" is easier for this problem than using ordinary integration with "top minus bottom". First, and most obvious reason is that solving E_1 explicitly for y in terms of x (or x in terms of y) is not a straight forward task. Even after writing both E_1 and E_2 as explicit functions of one variable, you will see that it takes multiple functions to represent the curves. This makes things much harder when deciding which piece(s) to use in the integral. Furthermore, even after completing that task and setting up the correct integral, you will notice that the bounds are super messy! If you are integrating with respect to x, then the lower bound will be $x = \sqrt{\frac{8+2\sqrt{2}}{7}}\cos\left(\frac{5\pi}{8}\right)$ and the upper bound will be $x = \sqrt{\frac{8-2\sqrt{2}}{7}}\cos\left(\frac{\pi}{8}\right)$.

Now if we use polar integration, then the bounds will be fairly simple. The lower bound will be $\theta = \frac{\pi}{8}$ and the upper bound will be $\theta = \frac{5\pi}{8}$. Furthermore, the integral requires r^2 which we already have both curves solved for.

Now let us actually set up the integral for the required region and solve it. Recall that the area bounded by two polar curves is given by:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta$$

where r_1 is the "outer" curve and r_2 is the "inner" curve.

The following diagram focuses on the desired area and it is clear that the "outer" curve is E_2 and the "inner" curve is E_1. This also confirms the lower bound to be $\theta_1 = \frac{\pi}{8}$ and the upper bound to be $\theta_2 = \frac{5\pi}{8}$.

Using $E_1 : r^2 = \frac{2}{2+\sin(2\theta)}$ and $E_2 : r^2 = \frac{2}{2+\sin\left(2\theta+\frac{\pi}{2}\right)}$, the integral representing the desired area between P and Q is

$$\begin{aligned} A &= \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2}\left(\frac{2}{2+\sin\left(2\theta+\frac{\pi}{2}\right)} - \frac{2}{2+\sin(2\theta)}\right) d\theta \\ &= \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2+\cos(2\theta)} d\theta - \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2+\sin(2\theta)} d\theta \qquad \left(\text{since } \sin\left(x+\frac{\pi}{2}\right) = \cos(x)\right) \end{aligned}$$

Now let us compute the following two integrals separately

$$A_1 = \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2+\cos(2\theta)} d\theta, \quad A_2 = \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2+\sin(2\theta)} d\theta$$

Computing A_1:

$$\begin{aligned}
A_1 &= \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2+\cos(2\theta)} d\theta \\
&= \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2+\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)} d\theta && (\text{since } \cos(2x) = \frac{1-\tan^2\theta}{1+\tan^2\theta}) \\
&= \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1+\tan^2(\theta)}{3+\tan^2(\theta)} d\theta \\
&= \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{\sec^2(\theta)}{3+\tan^2(\theta)} d\theta && (\text{since } \sec^2(x) = 1+\tan^2(x)) \\
&= \int_{\tan(\frac{\pi}{8})}^{\tan(\frac{5\pi}{8})} \frac{1}{3+u^2} du && (\text{let } u = \tan\theta, du = \sec^2\theta d\theta) \\
&= \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\right]_{\tan(\frac{\pi}{8})}^{\tan(\frac{5\pi}{8})} \\
&= \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{\tan\theta}{\sqrt{3}}\right)\right]_{\frac{\pi}{8}}^{\frac{5\pi}{8}} && (\text{returning to variable } \theta \text{ since } u = \tan\theta)
\end{aligned}$$

now we take into account the singularity at $\theta = \frac{\pi}{2}$ for $u = \tan\theta$,

$$\begin{aligned}
&\implies \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{\tan\theta}{\sqrt{3}}\right)\right]_{\frac{\pi}{8}}^{\frac{\pi}{2}} + \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{\tan\theta}{\sqrt{3}}\right)\right]_{\frac{\pi}{2}}^{\frac{5\pi}{8}} \\
&= \lim_{b\to\frac{\pi}{2}^-} \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{\tan\theta}{\sqrt{3}}\right)\right]_{\frac{\pi}{8}}^{b} + \lim_{b\to\frac{\pi}{2}^+} \left[\frac{1}{\sqrt{3}} \arctan\left(\frac{\tan\theta}{\sqrt{3}}\right)\right]_{b}^{\frac{5\pi}{8}} \\
&= \left(\frac{\pi}{2\sqrt{3}} - \frac{1}{\sqrt{3}} \arctan\left(\frac{\tan(\frac{\pi}{8})}{\sqrt{3}}\right)\right) + \left(\frac{1}{\sqrt{3}} \arctan\left(\frac{\tan(\frac{5\pi}{8})}{\sqrt{3}}\right) - \left(-\frac{\pi}{2\sqrt{3}}\right)\right) \\
&= \frac{\pi}{\sqrt{3}} + \frac{1}{\sqrt{3}} \arctan\left(\frac{\tan(\frac{5\pi}{8})}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}} \arctan\left(\frac{\tan(\frac{\pi}{8})}{\sqrt{3}}\right) \\
&= \frac{\pi - \arctan(\sqrt{6})}{\sqrt{3}} && (\text{used Wolfram Alpha to simplify}) \\
&\approx 1.1307
\end{aligned}$$

Computing A_2:

$$\begin{aligned}
A_2 &= \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2+\sin(2\theta)} d\theta \\
&= \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2+\sin(x)} dx && (\text{let } x = 2\theta, dx = 2d\theta)
\end{aligned}$$

Now use Weierstrass substitution. Let $t = \tan\left(\frac{x}{2}\right), \sin(x) = \frac{2t}{1+t^2}, dx = \frac{2}{1+t^2}dt$,

$$\begin{aligned}
&\implies \frac{1}{2}\int_{\tan(\frac{\pi}{8})}^{\tan(\frac{5\pi}{8})} \frac{1}{2+\left(\frac{2t}{1+t^2}\right)}\left(\frac{2}{1+t^2}dt\right) \\
&= \int_{\tan(\frac{\pi}{8})}^{\tan(\frac{5\pi}{8})} \frac{1}{2+2t^2+2t}dt \\
&= \frac{1}{2}\int_{\tan(\frac{\pi}{8})}^{\tan(\frac{5\pi}{8})} \frac{1}{t^2+t+1}dt \\
&= \frac{1}{2}\int_{\tan(\frac{\pi}{8})}^{\tan(\frac{5\pi}{8})} \frac{1}{(t+\frac{1}{2})^2+\frac{3}{4}}dt \qquad \text{(completed the square)} \\
&= \frac{2}{3}\int_{\tan(\frac{\pi}{8})}^{\tan(\frac{5\pi}{8})} \frac{1}{(\frac{2}{\sqrt{3}}t+\frac{\sqrt{3}}{3})^2+1}dt \\
&= \frac{\sqrt{3}}{3}\int_{\frac{2}{\sqrt{3}}\tan(\frac{\pi}{8})+\frac{\sqrt{3}}{3}}^{\frac{2}{\sqrt{3}}\tan(\frac{5\pi}{8})+\frac{\sqrt{3}}{3}} \frac{1}{v^2+1}dv \qquad \left(\text{let } v = \frac{2}{\sqrt{3}}t+\frac{\sqrt{3}}{3}, dv = \frac{2}{\sqrt{3}}dt\right) \\
&= \frac{\sqrt{3}}{3}\left[\arctan(v)\right]_{\frac{2}{\sqrt{3}}\tan(\frac{\pi}{8})+\frac{\sqrt{3}}{3}}^{\frac{2}{\sqrt{3}}\tan(\frac{5\pi}{8})+\frac{\sqrt{3}}{3}} \\
&= \frac{\sqrt{3}}{3}\left[\arctan\left(\frac{2}{\sqrt{3}}t+\frac{\sqrt{3}}{3}\right)\right]_{\tan(\frac{\pi}{8})}^{\tan(\frac{5\pi}{8})} \\
&= \frac{\sqrt{3}}{3}\left[\arctan\left(\frac{2}{\sqrt{3}}\tan\left(\frac{x}{2}\right)+\frac{\sqrt{3}}{3}\right)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\
&= \frac{\sqrt{3}}{3}\left[\arctan\left(\frac{2}{\sqrt{3}}\tan(\theta)+\frac{\sqrt{3}}{3}\right)\right]_{\frac{\pi}{8}}^{\frac{5\pi}{8}}
\end{aligned}$$

once again, we need to take into account the singularity at $x = \pi$ for $t = \tan\left(\frac{x}{2}\right)$,

$$\begin{aligned}
&\implies \frac{\sqrt{3}}{3}\left[\arctan\left(\frac{2}{\sqrt{3}}\tan(\theta)+\frac{\sqrt{3}}{3}\right)\right]_{\frac{\pi}{8}}^{\frac{\pi}{2}} + \frac{\sqrt{3}}{3}\left[\arctan\left(\frac{2}{\sqrt{3}}\tan(\theta)+\frac{\sqrt{3}}{3}\right)\right]_{\frac{\pi}{2}}^{\frac{5\pi}{8}} \\
&= \lim_{b\to\frac{\pi}{2}^-}\frac{\sqrt{3}}{3}\left[\arctan\left(\frac{2}{\sqrt{3}}\tan(\theta)+\frac{\sqrt{3}}{3}\right)\right]_{\frac{\pi}{8}}^{b} + \lim_{b\to\frac{\pi}{2}^+}\frac{\sqrt{3}}{3}\left[\arctan\left(\frac{2}{\sqrt{3}}\tan(\theta)+\frac{\sqrt{3}}{3}\right)\right]_{b}^{\frac{5\pi}{8}} \\
&= \frac{\sqrt{3}}{3}\left(\frac{\pi}{2}-\arctan\left(\frac{2}{\sqrt{3}}\tan\left(\frac{\pi}{8}\right)+\frac{\sqrt{3}}{3}\right)\right) + \frac{\sqrt{3}}{3}\left(\arctan\left(\frac{2}{\sqrt{3}}\tan\left(\frac{5\pi}{8}\right)+\frac{\sqrt{3}}{3}\right)-\left(-\frac{\pi}{2}\right)\right) \\
&= \frac{\sqrt{3}}{3}\pi + \frac{\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\tan\left(\frac{5\pi}{8}\right)+\frac{\sqrt{3}}{3}\right) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\tan\left(\frac{\pi}{8}\right)+\frac{\sqrt{3}}{3}\right) \\
&= \frac{\arctan(\sqrt{6})}{\sqrt{3}} \qquad \text{(used Wolfram Alpha to simplify)} \\
&\approx 0.68312
\end{aligned}$$

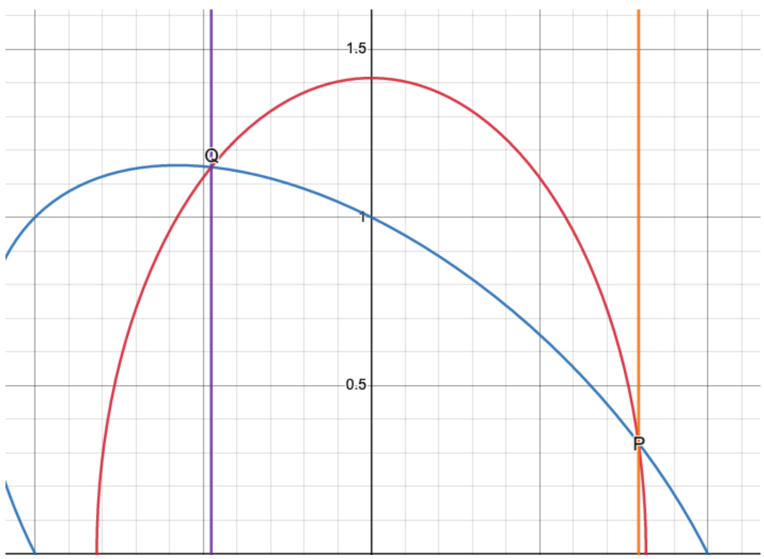
Overall, we have

$$\begin{aligned} A &= A_1 - A_2 \\ &= \frac{\pi - \arctan(\sqrt{6})}{\sqrt{3}} - \frac{\arctan(\sqrt{6})}{\sqrt{3}} \\ &= \frac{\pi - 2\arctan(\sqrt{6})}{\sqrt{3}} \\ &\approx 0.44756 \end{aligned}$$

Thus, the area of the region outside of E_1 and inside of E_2 is $\frac{\pi - 2\arctan(\sqrt{6})}{\sqrt{3}}$.

Extra

To check our work we can calculate the area using Cartesian integration. Let's set up the integral with respect to x using the following diagram:



The red curve is E_2 and the blue curve is E_1. We also found the Cartesian coordinates of P and Q in Part B so we can use their x-coordinates as the bounds of our integral. Last thing we require is to have the equations of both E_1 and E_2 solved for y in terms of x.

E_1: To solve for y in $x^2 + xy + y^2 = 1$, we use the quadratic formula and treat the equation as a quadratic in y:

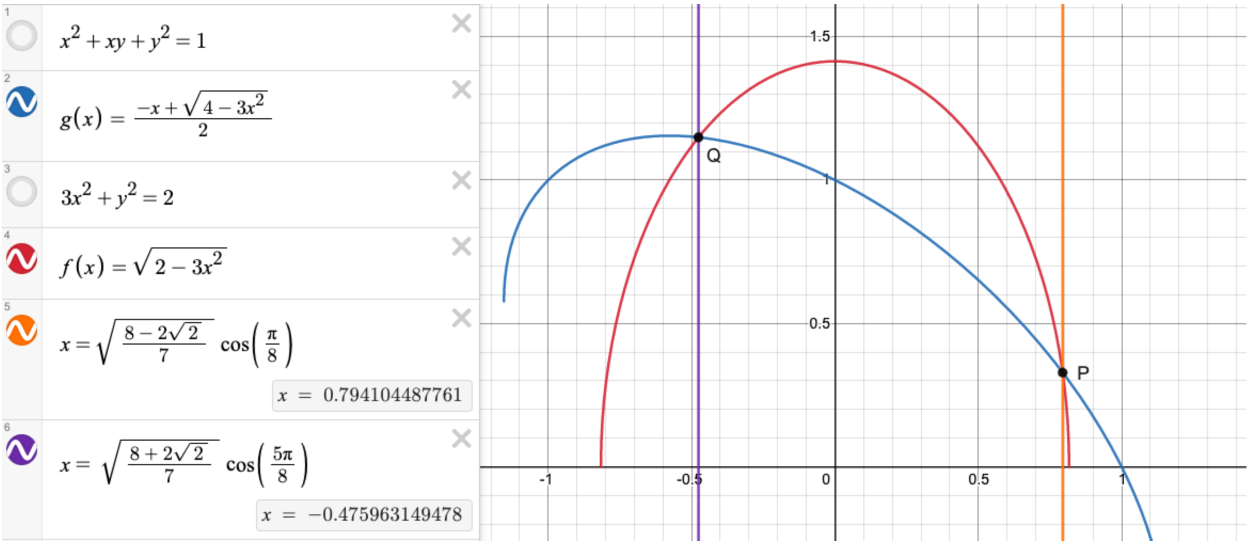
$$\begin{aligned} &\implies y^2 + (x)y + (x^2 - 1) = 0 \\ y &= \frac{-x \pm \sqrt{4 - 3x^2}}{2} \end{aligned}$$

E_2: To solve for y in $3x^2 + y^2 = 2$, simply isolate for it:

$$\implies y = \pm\sqrt{2 - 3x^2}$$

As we can see, both E_1 and E_2 have two functions representing them. In our integral, we need the positive pieces.

We can now set up the integral for the desired area very easily using this diagram:



On our interval, between the points Q and P, E_2 is always the "upper" curve represented by $f(x) = \sqrt{2-3x^2}$ and E_1 is the lower curve represented by $g(x) = \frac{-x+\sqrt{4-3x^2}}{2}$. The lower bound is $x = \sqrt{\frac{8+2\sqrt{2}}{7}}\cos(\frac{5\pi}{8})$ and the upper bound is $x = \sqrt{\frac{8-2\sqrt{2}}{7}}\cos(\frac{\pi}{8})$.

So the area of the desired region is:

$$A = \int_{\sqrt{\frac{8+2\sqrt{2}}{7}}\cos(\frac{5\pi}{8})}^{\sqrt{\frac{8-2\sqrt{2}}{7}}\cos(\frac{\pi}{8})} \left(\sqrt{2-3x^2} - \frac{-x+\sqrt{4-3x^2}}{2}\right) dx$$

Using a graphing calculator to evaluate the integral, we see that we get the same answer that we got above using polar integration:

$$\int_{\sqrt{\frac{8+2\sqrt{2}}{7}}\cos\left(\frac{5\pi}{8}\right)}^{\sqrt{\frac{8-2\sqrt{2}}{7}}\cos\left(\frac{\pi}{8}\right)} \left(\sqrt{2-3x^2} - \frac{-x+\sqrt{4-3x^2}}{2}\right) dx$$

$$= 0.447558102755$$