Problem 2: S P A C E V O Y A G E R 3 0 2 1

Part A

The error in Boppo's reasoning occurs when he tires to find the surface area. He should be adding the tangent vectors instead of subtracting them! He wants the total surface area not the difference of surface areas between the two surfaces.
Now the reason we subtract the two functions in the integrand for volume is because we can get the desired bounded volume by finding the volume under the larger surface and subtracting from it the volume under the smaller surface. This idea does not apply when we are trying to find the total surface area. We should simply find the surface area of each surface and add the results.

Part B

The space voyager is the solid bounded between the surfaces $z = f(x,y) = 1 - |x^2 - y|$ and $z = g(x,y) = x^2$. To find the volume of the space voyager, we first need to find the domain of integration E. To find E, we need the curve(s) of intersection that will bound E. Thus, we solve

$$\begin{aligned} f(x,y) &= g(x,y) \\ 1 - |x^2 - y| &= x^2 \\ 1 - x^2 &= |x^2 - y| \end{aligned}$$

We now have two cases,

$$\begin{cases} 1 - x^2 &= x^2 - y \quad (1) \\ 1 - x^2 &= y - x^2 \quad (2) \end{cases}$$

i) solving (1)

$$\begin{aligned} 1 - x^2 &= x^2 - y \\ 1 + y &= 2x^2 \\ y &= 2x^2 - 1 \end{aligned}$$

i) solving (2)

$$\begin{aligned} 1 - x^2 &= y - x^2 \\ y &= 1 \end{aligned}$$

So the boundary of E is given by the curves $y = 2x^2 - 1$ and $y = 1$. The region E is shown below,

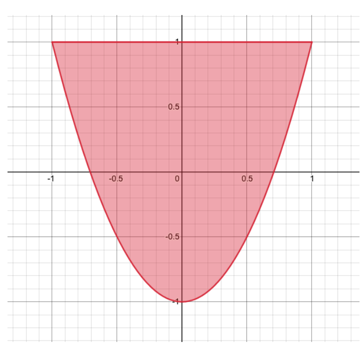


Figure 2: The region E

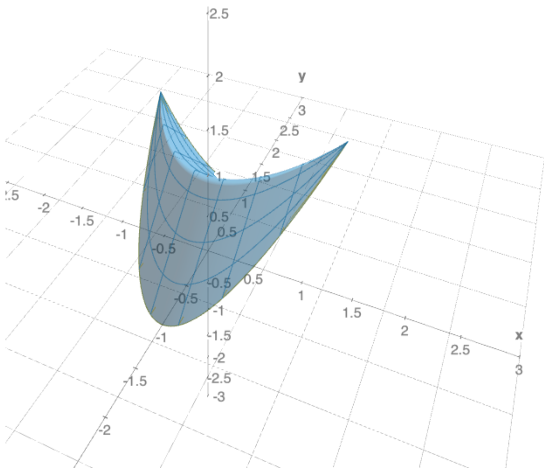
Now E can be described as both a Type I and Type II region,

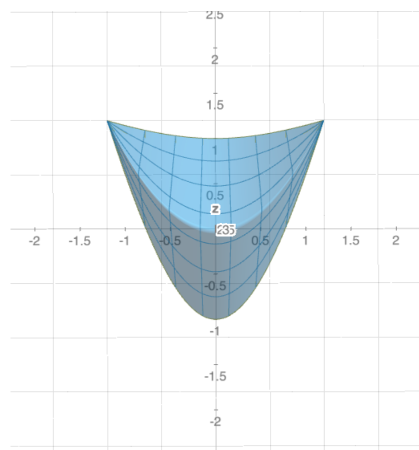
$$E = \{-1 \leq x \leq 1,\, 2x^2 - 1 \leq y \leq 1\} \text{ (Type I)}$$

$$E = \left\{ -\sqrt{\frac{1}{2}(y+1)} \leq x \leq \sqrt{\frac{1}{2}(y+1)},\, -1 \leq y \leq 1 \right\} \text{ (Type II)}$$

but describing it as a Type I region might make the computations easier.

Also note that over E, $f(x, y) \geq g(x, y)$ and that the desired volume is symmetric with respect to the plane $x = 0$:





thus we can integrate over $D = \{0 \leq x \leq 1,\, 2x^2 - 1 \leq y \leq 1\}$ and multiply the result by 2.

Now the volume of the space voyager is represented by the following double integral:

$$\begin{aligned} V &= 2\iint_D f(x,y)\, dA - 2\iint_D g(x,y)\, dA \\ &= 2\iint_D f(x,y) - g(x,y)\, dA \\ &= 2\int_0^1 \int_{2x^2-1}^1 1 - |x^2 - y| - x^2\, dy\, dx \end{aligned}$$

To evaluate this integral we need to consider cases that depend on the sign of $x^2 - y$.

$$|x^2 - y| = \begin{cases} x^2 - y & \text{if } y \leq x^2 \\ y - x^2 & \text{if } y > x^2 \end{cases}$$

Let us add this information to our region D and split the region into cases in order to evaluate the double integral.

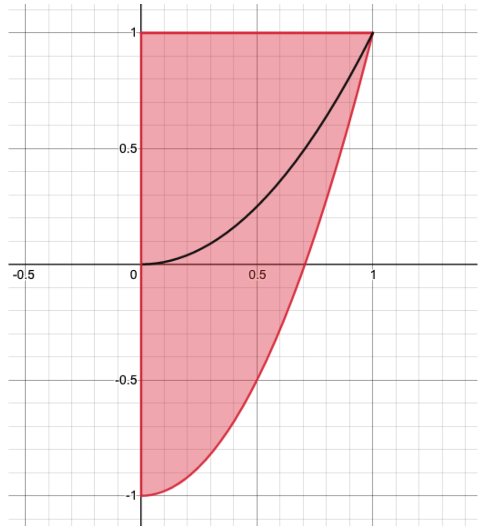


Figure 3: The region D

We need to split D depending on whether $y \leq x^2$ or $y > x^2$. Thus,

$$\begin{aligned} D &= D_1 \cup D_2 \\ &= \{0 \leq x \leq 1,\, 2x^2 - 1 \leq y \leq x^2\} \cup \{0 \leq x \leq 1,\, x^2 \leq y \leq 1\} \end{aligned}$$

where both D_1 and D_2 are Type I regions.

Finally, we can work on the integral:

$$\begin{aligned} V &= 2 \iint_D f(x,y) - g(x,y)\, dA \\ &= 2 \left(\int_0^1 \int_{2x^2-1}^{x^2} 1 - (x^2 - y) - x^2 \, dy\, dx + \int_0^1 \int_{x^2}^{1} 1 - (y - x^2) - x^2 \, dy\, dx \right) \\ &= 2 \int_0^1 \int_{2x^2-1}^{x^2} 1 - 2x^2 + y \, dy\, dx + 2 \int_0^1 \int_{x^2}^{1} 1 - y \, dy\, dx \end{aligned}$$

We can evaluate these integrals separately.

i) Evaluating the first integral:

$$
\begin{aligned}
2\int_0^1\int_{2x^2-1}^{x^2} 1-2x^2+y\,dy\,dx &= 2\int_0^1 \left[y-2x^2y+\frac{1}{2}y^2\right]_{2x^2-1}^{x^2} dx \\
&= 2\int_0^1 \left[x^2-2x^2(x^2)+\frac{1}{2}x^4-\left((2x^2-1)-2x^2(2x^2-1)+\frac{1}{2}(2x^2+1)^2\right)\right] dx \\
&= 2\int_0^1 \frac{1}{2}x^4-x^2+\frac{1}{2}\,dx \\
&= \int_0^1 x^4-2x^2+1\,dx \\
&= \left[\frac{1}{5}x^5-\frac{2}{3}x^3+x\right]_0^1 \\
&= \frac{8}{15}
\end{aligned}
$$

ii) Evaluating the second integral:

$$
\begin{aligned}
2\int_0^1\int_{x^2}^{1} 1-y\,dy\,dx &= 2\int_0^1 \left[y-\frac{1}{2}y^2\right]_{x^2}^{1} dx \\
&= 2\int_0^1 \left[1-\frac{1}{2}-\left(x^2-\frac{1}{2}x^4\right)\right] dx \\
&= 2\int_0^1 \frac{1}{2}x^4-x^2+\frac{1}{2}\,dx \\
&= 2\left[\frac{1}{10}x^5-\frac{1}{3}x^3+\frac{1}{2}x\right]_0^1 \\
&= \frac{8}{15}
\end{aligned}
$$

Overall we have,

$$
\begin{aligned}
V &= 2\iint_D f(x,y)-g(x,y)\,dA \\
&= 2\int_0^1\int_{2x^2-1}^{x^2} 1-2x^2+y\,dy\,dx + 2\int_0^1\int_{x^2}^{1} 1-y\,dy\,dx \\
&= \frac{8}{15}+\frac{8}{15} \\
&= \frac{16}{15}
\end{aligned}
$$

Therefore, the volume of the space voyager is exactly $\frac{16}{15}$ units^3.

Part C

Now let's find the surface area of the space voyager. This will be the sum of the surface areas of $f(x,y)$ and $g(x,y)$ over E. Again we can use symmetry and integrate over D. Thus we can write,

$$
\begin{aligned}
S &= 2\iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA + 2\iint_D \sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dA \\
&= 2\iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} + \sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dA
\end{aligned}
$$

Let us now find the first order partial derivatives of $f(x,y)$ and $g(x,y)$.

i) Partial derivatives of $f(x,y)$

$$
\begin{aligned}
f_x(x,y) &= \frac{\partial}{\partial x}\left(1-\left|x^2-y\right|\right) \\
&= \frac{\partial}{\partial x}1 - \frac{\partial}{\partial x}\sqrt{(x^2-y)^2} \\
&= 0 - \frac{1}{2\sqrt{(x^2-y)^2}}\cdot 2(x^2-y)\cdot 2x \\
&= -\frac{2x(x^2-y)}{|x^2-y|}
\end{aligned}
$$

$$
\begin{aligned}
f_y(x,y) &= \frac{\partial}{\partial y}\left(1-\left|x^2-y\right|\right) \\
&= \frac{\partial}{\partial y}1 - \frac{\partial}{\partial y}\sqrt{(x^2-y)^2} \\
&= 0 - \frac{1}{2\sqrt{(x^2-y)^2}}\cdot 2(x^2-y)\cdot -1 \\
&= \frac{x^2-y}{|x^2-y|}
\end{aligned}
$$

ii) Partial derivatives of $g(x,y)$

$$
\begin{aligned}
g_x(x,y) &= \frac{\partial}{\partial x}x^2 \\
&= 2x
\end{aligned}
$$

$$
\begin{aligned}
g_y(x,y) &= \frac{\partial}{\partial y}x^2 \\
&= 0
\end{aligned}
$$

So now we have,

$$
\begin{aligned}
S &= 2\iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}+\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\,dA \\
&= 2\iint_D \sqrt{1+\left(-\frac{2x(x^2-y)}{|x^2-y|}\right)^2+\left(\frac{x^2-y}{|x^2-y|}\right)^2}+\sqrt{1+(2x)^2+(0)^2}\,dA \\
&= 2\iint_D \sqrt{1+\frac{4x^2(x^2-y)^2}{(x^2-y)^2}+\frac{(x^2-y)^2}{(x^2-y)^2}}+\sqrt{1+4x^2}\,dA \\
&= 2\iint_D \sqrt{2+4x^2}+\sqrt{1+4x^2}\,dA \\
&= 4\iint_D \sqrt{\frac{1}{2}+x^2}\,dA+4\iint_D \sqrt{\frac{1}{4}+x^2}\,dA
\end{aligned}
$$

Let

$$I_1 = 4\iint_D \sqrt{\frac{1}{2}+x^2}\,dA$$

and

$$I_2 = 4\iint_D \sqrt{\frac{1}{4}+x^2}\,dA$$

Let's evaluate each integral separately over $D = \{0 \le x \le 1,\, 2x^2-1 \le y \le 1\}$.

i) Evaluating I_1 over D:

$$
\begin{aligned}
I_1 = 4\iint_D \sqrt{\frac{1}{2}+x^2}\,dA &= 4\int_0^1\int_{2x^2-1}^1 \sqrt{\frac{1}{2}+x^2}\,dy\,dx \\
&= 4\int_0^1 \left[\sqrt{\frac{1}{2}+x^2}\,y\right]_{2x^2-1}^1 dx \\
&= 4\int_0^1 \left[\sqrt{\frac{1}{2}+x^2}-\sqrt{\frac{1}{2}+x^2}\,(2x^2-1)\right] dx \\
&= 4\int_0^1 2\sqrt{\frac{1}{2}+x^2}-2x^2\sqrt{\frac{1}{2}+x^2}\,dx \\
&= 8\int_0^1 \sqrt{\frac{1}{2}+x^2}\,dx-8\int_0^1 x^2\sqrt{\frac{1}{2}+x^2}\,dx
\end{aligned}
$$

Let's evaluate these two integrals separately. Let

$$A_1 = 8\int_0^1 \sqrt{\frac{1}{2}+x^2}\,dx$$

and

$$A_2 = 8\int_0^1 x^2\sqrt{\frac{1}{2}+x^2}\,dx$$

For both these integrals we will do a trigonometric substitution with tangent.

$$\text{Let } x = \frac{1}{\sqrt{2}}\tan\theta,\ dx = \frac{1}{\sqrt{2}}\sec^2\theta\, d\theta \ \text{ then } \ \theta_1 = 0,\ \theta_2 = \arctan(\sqrt{2})$$

Evaluating A_1:

$$\begin{aligned}
A_1 = 8\int_0^1 \sqrt{\frac{1}{2}+x^2}\,dx &= 8\int_0^{\arctan(\sqrt{2})} \sqrt{\frac{1}{2}+\left(\frac{1}{\sqrt{2}}\tan\theta\right)^2}\,\frac{1}{\sqrt{2}}\sec^2\theta\,d\theta && (1)\\
&= 4\int_0^{\arctan(\sqrt{2})} \sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta && (2)\\
&= 4\int_0^{\arctan(\sqrt{2})} \sec^3\theta\,d\theta && (3)\\
&= 4\left[\frac{1}{2}\left(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\right)\right]_0^{\arctan(\sqrt{2})} && (4)\\
&= 2\left[\sqrt{6}+\ln(\sqrt{2}+\sqrt{3})\right] && (5)
\end{aligned}$$

Notes:

(1) Used the tangent substitution mentioned
(2) Factored and simplified
(3) Used $1+\tan^2\theta=\sec^2\theta$ and the fact that $\sec\theta>0$ on $\theta\in\left[0,\arctan\left(\sqrt{2}\right)\right]$
(4) Used the antiderivative of $\sec^3\theta$. The work for it is shown at the end of the problem.
(5) Used WolframAlpha to simplify

Evaluating A_2:

$$\begin{aligned}
A_2 = 8\int_0^1 x^2\sqrt{\frac{1}{2}+x^2}\,dx &= 8\int_0^{\arctan(\sqrt{2})} \left(\frac{1}{\sqrt{2}}\tan\theta\right)^2\sqrt{\frac{1}{2}+\left(\frac{1}{\sqrt{2}}\tan\theta\right)^2}\,\frac{1}{\sqrt{2}}\sec^2\theta\,d\theta && (1)\\
&= 2\int_0^{\arctan(\sqrt{2})} \tan^2\theta\sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta && (2)\\
&= 2\int_0^{\arctan(\sqrt{2})} \tan^2\theta\sec^3\theta\,d\theta && (3)\\
&= 2\int_0^{\arctan(\sqrt{2})} \sec^5\theta-\sec^3\theta\,d\theta\\
&= 2\left[-\frac{\ln|\tan\theta+\sec\theta|}{8}+\frac{\sec^3\theta\tan\theta}{4}-\frac{\sec\theta\tan\theta}{8}\right]_0^{\arctan(\sqrt{2})} && (4)\\
&= 2\left[-\frac{1}{8}\ln(\sqrt{2}+\sqrt{3})+\frac{3\sqrt{6}}{4}-\frac{\sqrt{6}}{8}\right] && (5)\\
&= \frac{1}{4}\left[5\sqrt{6}-\ln(\sqrt{2}+\sqrt{3})\right]
\end{aligned}$$

Notes:

(1) Used the tangent substitution mentioned
(2) Factored and simplified
(3) Used $1 + \tan^2\theta = \sec^2\theta$ and the fact that $\sec\theta > 0$ on $\theta \in \left[0, \arctan\left(\sqrt{2}\right)\right]$
(4) Used the antiderivative of $\sec^5\theta$ and $\sec^3\theta$. The work for it is shown at the end of the problem.
(5) Used WolframAlpha to simplify

Overall we have,

$$\begin{aligned} I_1 &= A_1 - A_2 \\ 4\iint_D \sqrt{\frac{1}{2} + x^2}\,dA &= 8\int_0^1 \sqrt{\frac{1}{2} + x^2}\,dx - 8\int_0^1 x^2\sqrt{\frac{1}{2} + x^2}\,dx \\ &= 2\left[\sqrt{6} + \ln(\sqrt{2} + \sqrt{3})\right] - \frac{1}{4}\left[5\sqrt{6} - \ln(\sqrt{2} + \sqrt{3})\right] \\ &= \frac{3}{4}\left[\sqrt{6} + 3\ln\left(\sqrt{2} + \sqrt{3}\right)\right] \end{aligned}$$

ii) Similarly, evaluating I_2 over D:

$$\begin{aligned} I_2 = 4\iint_D \sqrt{\frac{1}{4} + x^2}\,dA &= 4\int_0^1 \int_{2x^2-1}^1 \sqrt{\frac{1}{4} + x^2}\,dy\,dx \\ &= 4\int_0^1 \left[\sqrt{\frac{1}{4} + x^2}\,y\right]_{2x^2-1}^1 dx \\ &= 4\int_0^1 \left[\sqrt{\frac{1}{4} + x^2} - \sqrt{\frac{1}{4} + x^2}\,(2x^2 - 1)\right] dx \\ &= 4\int_0^1 2\sqrt{\frac{1}{4} + x^2} - 2x^2\sqrt{\frac{1}{4} + x^2}\,dx \\ &= 8\int_0^1 \sqrt{\frac{1}{4} + x^2}\,dx - 8\int_0^1 x^2\sqrt{\frac{1}{4} + x^2}\,dx \end{aligned}$$

Let's evaluate these two integrals separately. Let

$$B_1 = 8\int_0^1 \sqrt{\frac{1}{4} + x^2}\,dx$$

and

$$B_2 = 8\int_0^1 x^2\sqrt{\frac{1}{4} + x^2}\,dx$$

For both these integrals we will do a trigonometric substitution with tangent.

$$\text{Let } x = \frac{1}{2}\tan\theta,\ dx = \frac{1}{2}\sec^2\theta\,d\theta \ \text{ then } \ \theta_1 = 0,\ \theta_2 = \arctan(2)$$

Evaluating B_1:

$$B_1 = 8\int_0^1 \sqrt{\frac{1}{4} + x^2}\,dx = 8\int_0^{\arctan(2)} \sqrt{\frac{1}{4} + \left(\frac{1}{2}\tan\theta\right)^2}\,\frac{1}{2}\sec^2\theta\,d\theta \tag{1}$$

$$= 2\int_0^{\arctan(2)} \sqrt{1 + \tan^2\theta}\,\sec^2\theta\,d\theta \tag{2}$$

$$= 2\int_0^{\arctan(2)} \sec^3\theta\,d\theta \tag{3}$$

$$= 2\left[\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right]_0^{\arctan(2)} \tag{4}$$

$$= \left[2\sqrt{5} + \ln(2 + \sqrt{5}) + (0 - 0)\right] \tag{5}$$

$$= 2\sqrt{5} + \ln(2 + \sqrt{5})$$

Notes:

(1) Used the tangent substitution mentioned
(2) Factored and simplified
(3) Used $1 + \tan^2\theta = \sec^2\theta$ and the fact that $\sec\theta > 0$ on $\theta \in [0, \arctan(2)]$
(4) Used the antiderivative of $\sec^3\theta$. The work for it is shown at the end of the problem.
(5) Used WolframAlpha to simplify

Evaluating B_2:

$$B_2 = 8\int_0^1 x^2\sqrt{\frac{1}{4} + x^2}\,dx = 8\int_0^{\arctan(2)} \left(\frac{1}{2}\tan\theta\right)^2 \sqrt{\frac{1}{4} + \left(\frac{1}{2}\tan\theta\right)^2}\,\frac{1}{2}\sec^2\theta\,d\theta \tag{1}$$

$$= \frac{1}{2}\int_0^{\arctan(2)} \tan^2\theta\sqrt{1 + \tan^2\theta}\,\sec^2\theta\,d\theta \tag{2}$$

$$= \frac{1}{2}\int_0^{\arctan(2)} \tan^2\theta\sec^3\theta\,d\theta \tag{3}$$

$$= \frac{1}{2}\int_0^{\arctan(2)} \sec^5\theta - \sec^3\theta\,d\theta$$

$$= \frac{1}{2}\left[-\frac{\ln|\tan\theta + \sec\theta|}{8} + \frac{\sec^3\theta\tan\theta}{4} - \frac{\sec\theta\tan\theta}{8}\right]_0^{\arctan(2)} \tag{4}$$

$$= \frac{1}{2}\left[-\frac{1}{8}\ln(2 + \sqrt{5}) + \frac{5\sqrt{5}}{2} - \frac{\sqrt{5}}{4}\right] \tag{5}$$

$$= \frac{1}{16}\left[18\sqrt{5} - \ln(2 + \sqrt{5})\right]$$

Notes:

(1) Used the tangent substitution mentioned
(2) Factored and simplified
(3) Used $1+\tan^2\theta = \sec^2\theta$ and the fact that $\sec\theta > 0$ on $\theta \in [0, \arctan(2)]$
(4) Used the antiderivative of $\sec^5\theta$ and $\sec^3\theta$. The work for it is shown at the end of the problem.
(5) Used WolframAlpha to simplify

Overall we have,

$$\begin{aligned} I_2 &= B_1 - B_2 \\ 4\iint_D \sqrt{\frac{1}{4}+x^2}\,dA &= 8\int_0^1 \sqrt{\frac{1}{4}+x^2}\,dx - 8\int_0^1 x^2\sqrt{\frac{1}{4}+x^2}\,dx \\ &= 2\sqrt{5} + \ln(2+\sqrt{5}) - \frac{1}{16}\left[18\sqrt{5} - \ln(2+\sqrt{5})\right] \\ &= \frac{1}{16}\left(14\sqrt{5} + 17\ln\left(2+\sqrt{5}\right)\right) \end{aligned}$$

Finally, we have

$$\begin{aligned} S &= I_1 + I_2 \\ &= 4\iint_D \sqrt{\frac{1}{2}+x^2}\,dA + 4\iint_D \sqrt{\frac{1}{4}+x^2}\,dA \\ &= \frac{3}{4}\left[\sqrt{6} + 3\ln\left(\sqrt{2}+\sqrt{3}\right)\right] + \frac{1}{16}\left(14\sqrt{5} + 17\ln\left(2+\sqrt{5}\right)\right) \\ &= \frac{1}{16}\left(14\sqrt{5} + 12\sqrt{6} + 36\ln(\sqrt{2}+\sqrt{3}) + 17\ln(2+\sqrt{5})\right) \\ &\approx 7.9065 \end{aligned}$$

Therefore, the exact surface area of the space voyager is

$$\frac{1}{16}\left(14\sqrt{5} + 12\sqrt{6} + 36\ln(\sqrt{2}+\sqrt{3}) + 17\ln(2+\sqrt{5})\right)\text{ units}^2$$

which is approximately 7.9065units^2.

Here is the work for the integrals of $\sec^3\theta$ and $\sec^5\theta$. Let $I_1 = \int \sec^3\theta \, d\theta$ and let $I_2 = \int \sec^5\theta \, d\theta$.

i) Evaluate I_1:

$$\int \sec^3\theta \, d\theta = \int \sec\theta \cdot \sec^2\theta \, d\theta$$

Use Integration by Parts with

$$u = \sec\theta, \; dv = \sec^2\theta \, d\theta, \; v = \tan\theta, \; du = \sec\theta \tan\theta \, d\theta$$

Now we have,

$$\begin{aligned}
\int \sec^3\theta \, d\theta &= \sec\theta\tan\theta - \int \tan\theta \sec\theta \tan\theta \, d\theta \\
&= \sec\theta\tan\theta - \int \tan^2\theta \sec\theta \, d\theta \\
&= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta \, d\theta \\
&= \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta \, d\theta
\end{aligned}$$

Notice that we have I_1 appearing on both sides of the integral equation:

$$\begin{aligned}
\int \sec^3\theta \, d\theta &= \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta \\
I_1 &= \sec\theta\tan\theta - I_1 + \int \sec\theta \, d\theta
\end{aligned}$$

Thus, solving for I_1 gives:

$$\begin{aligned}
I_1 &= \sec\theta\tan\theta - I_1 + \int \sec\theta \, d\theta \\
2I_1 &= \sec\theta\tan\theta + \int \sec\theta \, d\theta \\
I_1 &= \frac{1}{2}\left(\sec\theta\tan\theta + \int \sec\theta \, d\theta\right)
\end{aligned}$$

Now substituting $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta|$ (this is a standard result) gives

$$I_1 = \int \sec^3\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)$$

ii) Similarly, evaluate I_2:

$$\int \sec^5\theta \, d\theta = \int \sec^3\theta \sec^2\theta \, d\theta$$

Use Integration by Parts with

$$u = \sec^3\theta, \; dv = \sec^2\theta \, d\theta, \; v = \tan\theta, \; du = 3\sec^2\theta \sec\theta \tan\theta \, d\theta$$

Now we have,

$$\begin{aligned}
\int \sec^5\theta\,d\theta &= \sec^3\theta\tan\theta - 3\int \tan\theta\sec^2\theta\sec\theta\tan\theta\,d\theta \\
&= \sec^3\theta\tan\theta - 3\int \tan^2\theta\sec^3\theta\,d\theta \\
&= \sec^3\theta\tan\theta - 3\int (\sec^2\theta - 1)\sec^3\theta\,d\theta \\
&= \sec^3\theta\tan\theta - 3\int \sec^5\theta - \sec^3\theta\,d\theta
\end{aligned}$$

Notice that we have I_2 appearing on both sides of the integral equation:

$$\begin{aligned}
\int \sec^5\theta\,d\theta &= \sec^3\theta\tan\theta - 3\int \sec^5\theta\,d\theta + 3\int \sec^3\theta\,d\theta \\
I_2 &= \sec^3\theta\tan\theta - 3I_2 + 3\int \sec^3\theta\,d\theta
\end{aligned}$$

Thus, solving for I_2 gives:

$$\begin{aligned}
I_2 &= \sec^3\theta\tan\theta - 3I_2 + 3\int \sec^3\theta\,d\theta \\
4I_2 &= \sec^3\theta\tan\theta + 3\int \sec^3\theta\,d\theta \\
I_2 &= \frac{1}{4}\left(\sec^3\theta\tan\theta + 3\int \sec^3\theta\,d\theta\right)
\end{aligned}$$

Now substituting $\int \sec^3\theta\,d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)$ (this is I_1) gives

$$\begin{aligned}
I_2 = \int \sec^5\theta &= \frac{1}{4}\left(\sec^3\theta\tan\theta + 3\left(\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)\right)\right) \\
&= \frac{\sec^3\theta\tan\theta}{4} + \frac{3\sec\theta\tan\theta}{8} + \frac{3\ln|\sec\theta + \tan\theta|}{8}
\end{aligned}$$

Note that the integrals we computed for **Part C** used

$$I_1 = \int \sec^3\theta\,d\theta = \frac{\sec\theta\tan\theta}{2} + \frac{\ln|\sec\theta + \tan\theta|}{2}$$

and

$$\begin{aligned}
I_3 = I_2 - I_1 &= \int \sec^5\theta\,d\theta - \int \sec^3\theta\,d\theta \\
&= \frac{\sec^3\theta\tan\theta}{4} + \frac{3\sec\theta\tan\theta}{8} + \frac{3\ln|\sec\theta + \tan\theta|}{8} - \left[\frac{\sec\theta\tan\theta}{2} + \frac{\ln|\sec\theta + \tan\theta|}{2}\right] \\
&= \frac{\sec^3\theta\tan\theta}{4} + \frac{3\sec\theta\tan\theta}{8} + \frac{3\ln|\sec\theta + \tan\theta|}{8} - \frac{\sec\theta\tan\theta}{2} - \frac{\ln|\sec\theta + \tan\theta|}{2} \\
&= -\frac{\ln|\sec\theta + \tan\theta|}{8} + \frac{\sec^3\theta\tan\theta}{4} - \frac{\sec\theta\tan\theta}{8}
\end{aligned}$$